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LECTURES ON QUANTUM MECHANICS

VOLUME I

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CHAPTER XI.

RELATIVISTIC QUANTUM MECHANICS.

11·1 (1). *Conservation of Electric Charge.*

ACCORDING to modern ideas, an electric charge is essentially a quantity having discrete magnitude, which is determined by counting the number of electrons and protons. The total number of electrons and protons cannot naturally depend on the motion of the observers. We must therefore have the electric charge invariant for all observers.

Starting from this assumption of the invariance of electric charge, we shall demonstrate that the electro-magnetic field equations satisfy the principle of relativity. Thus we should have no reason to modify these equations.

11·1 (2). *Density of Electric Charge and Electric Current.*

First we shall find the laws of transformation for density and current. For definiteness we assume all electric charges to be composed of elementary charges, each of amount e.

In the neighbourhood of a specified point P of space, let $n(u)$ be the number of these charges per unit volume moving with velocity $\vec{u}$, as observed by S. Let n_0 be the number of the same charges per unit volume, as observed by S_0 who is moving with them. Then S sees this latter volume as contracted by a factor $\sqrt{1-\frac{u^2}{c^2}}$ in the direction of its motion, and therefore estimates the density as $\frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$ times greater than does S_0, so that we find

$$n(u) = \frac{n_0}{\sqrt{1-\frac{u^2}{c^2}}}. \tag{1}$$

In the frame K of the observer S, the charges under consideration will contribute an amount $en(u)$ to the charge-density ρ. The number of elementary charges which cross a unit area placed perpendicular to the x-axis per unit time is $u_x\, n(u)$, so that they

contribute an amount $e\, u_x\, n\,(u)$ to the x-component of the current-density $\vec{\sigma}$. Similarly for the other components. Hence in K, we have

$$\rho = \sum_u en\,(u), \quad \vec{\sigma} = \sum_u e\,\vec{u}\,n\,(u), \tag{2}$$

where the summation is taken over all the velocities u which occur.

Now consider an observer S′ at rest in a frame K′ which is moving relative to K with a velocity v in the direction of the positive x-axis. Then S′ obtains similarly for the charges considered in (1) a number-density $n'\,(u')$ given by

$$n'\,(u') = \frac{n_0}{\sqrt{1 - \frac{u'^2}{c^2}}}, \tag{3}$$

where u' is the velocity of the charges as observed by S′.

From the equations (3) and (5) of § 1·6 (1), we get the identity :

$$\frac{1}{\sqrt{1 - \frac{u'^2}{c^2}}} = \frac{\gamma\,(1 - u_x v/c^2)}{\sqrt{1 - \frac{u^2}{c^2}}}, \quad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{4}$$

Therefore on account of (1) and (3) we obtain

$$n'\,(u') = \gamma\,(1 - u_x v/c^2)\,\frac{n_0}{\sqrt{1 - \frac{u^2}{c^2}}}$$

$$= \gamma\left(1 - \frac{u_x v}{c^2}\right) n\,(u). \tag{5}$$

But from the Lorentz-transformation

$$dx' = \gamma\,(dx - vdt),\ dy' = dy,\ dz' = dz,\ dt' = \gamma\left(dt - \frac{v}{c^2}\,dx\right),$$

we find the law of transformation of velocity :

$$u_x' = \frac{dx'}{dt'} = \frac{\gamma\,(dx - v\,dt)}{\gamma\left(dt - \frac{v}{c^2}\,dx\right)} = \frac{u_x - v}{1 - \frac{u_x v}{c^2}}, \qquad \left[u_x = \frac{dx}{dt}\right],$$

$$u_y' = \frac{dy'}{dt'} = \frac{dy}{\gamma\left(dt - \frac{v}{c^2}\,dx\right)} = \frac{u_y}{\gamma\left(1 - \frac{u_x v}{c^2}\right)}, \quad \left[u_y = \frac{dy}{dt}\right],$$

$$u_z' = \frac{dz'}{dt'} = \frac{dz}{\gamma\left(dt - \frac{v}{c^2}\,dx\right)} = \frac{u_z}{\gamma\left(1 - \frac{u_x v}{c^2}\right)}, \quad \left[u_z = \frac{dz}{dt}\right]. \tag{6}$$

Substituting these in (5), we obtain:

$$u_x' n'(u') = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} \cdot \gamma \left(1 - \frac{u_x v}{c^2}\right) n(u) = \gamma (u_x - v)\, n(u),$$

$$u_y' n'(u') = u_y n(u), \; u_z' n'(u) = u_z n(u). \tag{7}$$

Hence, the charge and current densities in the frame K′ are given by

$$\rho' = \Sigma\, e n'(u') = \Sigma\, e\, \gamma \left(1 - \frac{u_x v}{c^2}\right) n(u)$$

$$= \gamma \left(\rho - \sigma_x \frac{v}{c^2}\right); \tag{8}$$

$$\sigma_x' = \Sigma\, e u_x' n'(u') = \Sigma\, e\gamma (u_x - v)\, n(u) = \gamma (\sigma_x - v\rho),$$

$$\sigma_y' = \Sigma\, e u_y' n'(u') = \Sigma\, e u_y n(u) = \sigma_y,$$

$$\sigma_z' = \Sigma\, e u_z' n'(u') = \Sigma\, e u_z n(u) = \sigma_z. \tag{9}$$

Thus the law of transformation for $(\sigma_x, \sigma_y, \sigma_z, \rho)$ is the same as the Lorentz-transformation for (x, y, z, t). Consequently, the inverse transformation would be

$$\sigma_x = \gamma (\sigma_x' + v\rho), \; \sigma_y = \sigma_y', \; \sigma_z = \sigma_z', \; \rho = \gamma \left(\rho' + \sigma_x' \frac{v}{c^2}\right). \tag{10}$$

11·1 (3). *The Maxwell-Lorentz Field Equations.*

From classical electrodynamics, we know that the Maxwell-Lorentz field equations are

$$\operatorname{div} \vec{E} = \rho, \quad \operatorname{div} \vec{H} = 0, \tag{1}$$

$$\operatorname{curl} \vec{E} = -\frac{1}{c} \frac{\partial \vec{H}}{\partial t}, \quad \operatorname{curl} \vec{H} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t} + \rho \frac{\vec{u}}{c}, \tag{2}$$

where $\vec{E}$ and $\vec{H}$ are the intensities of the electric and magnetic fields, ρ the density of the electric charge, and $\vec{u}$ the velocity with which it is moving.

From the equation (2) we obtain, since $\vec{\sigma} = \rho \vec{u}$,

$$\operatorname{div} \operatorname{curl} \vec{H} = \frac{1}{c} \frac{\partial}{\partial t} (\operatorname{div} \vec{E}) + \frac{1}{c} \operatorname{div} \vec{\sigma}.$$

But the divergence of a curl is identically zero, therefore substituting $\operatorname{div} \vec{E} = \rho$ from (1) we obtain:

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \vec{\sigma} \equiv \frac{\partial \rho}{\partial t} + \frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0. \tag{3}$$

This is the equation of continuity which expresses the conservation of total electric charge.

Further, if we consider the field equations for the case of free space with $\rho = 0$, then from (2) we get:

$$\text{curl curl } \vec{E} = -\frac{1}{c}\frac{\partial}{\partial t} \text{curl } \vec{H} = -\frac{1}{c}\frac{\partial}{\partial t}\left\{\frac{1}{c}\frac{\partial \vec{E}}{\partial t}\right\}.$$

But we know from vector analysis:

$$\text{curl curl } \vec{E} = \text{grad div } \vec{E} - \nabla^2 \vec{E}, \tag{4}$$

where ∇^2 is the Laplacian operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

From the last two equations we obtain, since $\text{div } \vec{E} = \rho = 0$,

$$-\nabla^2 \vec{E} = -\frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}, \text{ or } \Box \vec{E} = 0, \tag{5}$$

where $\Box$ is the D'Alumbertian $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$.

Similarly, for the magnetic field $\vec{H}$, we find

$$\Box \vec{H} \equiv \nabla^2 \vec{H} - \frac{1}{c^2}\frac{\partial^2 \vec{H}}{\partial t^2} = 0. \tag{6}$$

These are the well-known equations for the propagation of Hertzian waves in free space, with the velocity c.

In the general case, we know from electrodynamics that the fields $\vec{E}$ and $\vec{H}$ can be determined in terms of the scalar potential ϕ and the vector potential $\vec{A}$, by means of the equations

$$\vec{E} = -\text{grad } \phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}, \tag{7}$$

$$\vec{H} = \text{curl } \vec{A}. \tag{8}$$

But these equations do not determine the electromagnetic potentials A_x, A_y, A_z, ϕ uniquely. The potentials are concerned in the actual phenomena only through their curl, *viz.*, the electromagnetic force. The curl is unaltered if we replace $-A_x$ by $-A_x + \frac{\partial V}{\partial x}$, $-A_y$ by $-A_y + \frac{\partial V}{\partial y}$, $-A_z$ by $-A_z + \frac{\partial V}{\partial z}$, ϕ by $\phi + \frac{\partial V}{\partial t}$, where V is an arbitrary function of x, y, z, t. To avoid

arbitrariness, an additional condition is imposed on the electromagnetic potentials :

$$\operatorname{div} \vec{A} + \frac{1}{c}\frac{\partial \phi}{\partial t} \equiv \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} + \frac{1}{c}\frac{\partial \phi}{\partial t} = 0. \tag{9}$$

Taking the divergence of both sides in (7) we obtain :

$$\operatorname{div} \vec{E} = - \operatorname{div} \operatorname{grad} \phi - \frac{1}{c}\frac{\partial}{\partial t} (\operatorname{div} \vec{A}).$$

But we have $\operatorname{div} \operatorname{grad} \phi = \nabla^2 \phi$, $\operatorname{div} \vec{E} = \rho$, $\operatorname{div} \vec{A} = -\frac{1}{c}\frac{\partial \phi}{\partial t}$,

Therefore

$$\nabla^2 \phi - \frac{1}{\partial^2}\frac{\partial^2 \phi}{\partial t^2} = -\rho \text{ or } \Box\, \phi = -\rho. \tag{10}$$

From equation (8) we obtain :

$$\operatorname{curl} \vec{H} = \operatorname{curl} \operatorname{curl} \vec{A} = \operatorname{grad} \operatorname{div} \vec{A} - \nabla^2 \vec{A},$$

or

$$\frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{\vec{\sigma}}{c} = \operatorname{grad}\left(-\frac{1}{c}\frac{\partial \phi}{\partial t}\right) - \nabla^2 \vec{A}$$

$$= \frac{1}{c}\frac{\partial}{\partial t}\left(\vec{E} + \frac{1}{c}\frac{\partial \vec{A}}{\partial t}\right) - \nabla^2 \vec{A}.$$

Therefore

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\frac{\vec{\sigma}}{c} \text{ or } \Box\, \vec{A} = -\frac{1}{c}\vec{\sigma}. \tag{11}$$

We shall now transform the equations (10) and (11) to the frame K′. In § 1·3 we have shown that the operator $\Box$ is an invariant, so that we have $\Box' = \Box$. From the transformation equations (10) § 11·1 (2), we get therefore

$$\Box' A_x = -\frac{\gamma}{c}(\sigma_x' + v\rho'),\ \Box' A_y = -\frac{1}{c}\sigma_y',\ \Box' A_z = -\frac{1}{c}\sigma_z',$$

$$\Box' \phi = -\gamma\left(\rho' + \frac{\sigma_x' v}{c^2}\right). \tag{12}$$

Multiplying $\Box' \phi$ by $\gamma \frac{v}{c}$ and $\Box' A_x$ by γ, and subtracting the first from the second, we obtain

$$\Box' \gamma\left(A_x - \frac{v\phi}{c}\right) = -\frac{1}{c}\sigma_x'. \tag{13}$$

Similarly, multiplying $\Box' A_x$ by $\gamma \frac{v}{c}$ and $\Box \phi$ by γ, and subtracting the first from the second, we obtain

$$\Box' \gamma\left(\phi - \frac{v A_x}{c}\right) = -\rho'. \tag{14}$$

Hence the equation (12) may be written in the form

$$\Box' \vec{A}' = -\frac{1}{c}\vec{\sigma}', \ \Box' \phi' = -\rho', \tag{15}$$

provided we set

$$A'_x = \gamma\left(A_x - \frac{v\phi}{c}\right), \ A_y' = A_y, A_z' = A_z, \phi' = \gamma\left(\phi - \frac{v}{c}A_x\right). \tag{16}$$

This again gives the law of transformation for $\left(A_x, A_y, A_z, \frac{\phi}{c}\right)$ of the same form as that for (x, y, z, t). The inverse transformation will be therefore

$$A_x = \gamma\left(A_x + \frac{v\phi'}{c}\right), A_y = A_y', \ A_z = A_z', \phi = \gamma\left(\phi' + \frac{v}{c} A_x'\right). \tag{17}$$

Substituting these values, we find that the equation (9) is transformed to

$$\operatorname{div}' \vec{A}' + \frac{1}{c}\frac{\partial \phi'}{\partial t'} \equiv \frac{\partial A_x'}{\partial x'} + \frac{\partial A_y'}{\partial y'} + \frac{\partial A_z'}{\partial z'} + \frac{1}{c}\frac{\partial \phi'}{\partial t'} = 0. \tag{18}$$

From (15) and (18) we see that the field equations hold unchanged in form in the system K′, when the potentials are transformed according to the law (16).

We now form the quantities $\vec{E}'$, $\vec{H}'$, given by

$$\vec{E}' = -\operatorname{grad}' \phi' - \frac{1}{c}\frac{\partial \vec{A}'}{\partial t'}, \ \vec{H}' = \operatorname{curl}' \vec{A}'. \tag{19}$$

These $\vec{E}'$ and $\vec{H}'$ are the quantities recognised by S′ as the electric and magnetic intensities, because in his variables (x', y', z', t') they satisfy a set of equations in every respect of the same form as those satisfied by $\vec{E}$ and $\vec{H}$ for S.

The components of $\vec{E}'$ and $\vec{H}'$ can be evaluated by using (7), (8) and (16), together with the transformation equations for $\frac{\partial}{\partial x'}, \frac{\partial}{\partial y'}, \frac{\partial}{\partial z'}, \frac{\partial}{\partial t'}$. The latter are seen to be

$$\frac{\partial}{\partial x'} = \frac{\partial}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial}{\partial t}\frac{\partial t}{\partial x'} = \gamma\left(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t}\right), \frac{\partial}{\partial y'} = \frac{\partial}{\partial y}, \frac{\partial}{\partial z'} = \frac{\partial}{\partial z},$$

$$\frac{\partial}{\partial t'} = \frac{\partial}{\partial x}\frac{\partial x}{\partial t'} + \frac{\partial}{\partial t}\frac{\partial t}{\partial t'} = \gamma\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right).$$

Then from (19) we obtain:

$$E_x' = -\frac{\partial\phi'}{\partial x'} - \frac{1}{c}\frac{\partial A_x'}{\partial t'} = -\gamma\left(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t}\right)\phi' - \frac{1}{c}\gamma\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)A_x'$$

$$= -\gamma\left(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t}\right)\cdot\gamma\left(\phi - \frac{v}{c}A_x\right)$$

$$- \frac{1}{c}\gamma\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)\gamma\left(A_x - \frac{v}{c}\phi\right)$$

$$= -\frac{\partial\phi}{\partial x} - \frac{1}{c}\frac{\partial A_x}{\partial t} = E_x.$$

$$H_z' = \frac{\partial A_y'}{\partial x'} - \frac{\partial A_x'}{\partial y'} = \gamma\left(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t}\right)A_y - \frac{\partial}{\partial y}\cdot\gamma\left(A_x - \frac{v}{c}\phi\right)$$

$$= \gamma\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) + \gamma\frac{v}{c}\left(\frac{\partial\phi}{\partial y} + \frac{1}{c}\frac{\partial A_y}{\partial t}\right)$$

$$= \gamma\left(H_z - \frac{v}{c}E_y\right).$$

In this way, we find the law of transformation of $\vec{E}$ and $\vec{H}$ as follows:

$$\left.\begin{aligned} E_x' &= E_x,\ E_y' = \gamma\left(E_y - \frac{v}{c}H_z\right),\ E_z' = \gamma\left(E_z + \frac{v}{c}H_y\right);\\ H_x' &= H_x,\ H_y' = \gamma\left(H_y + \frac{v}{c}E_z\right),\ H_z' = \gamma\left(H_z - \frac{v}{c}E_y\right). \end{aligned}\right\} \quad (20)$$

The inverse transformation is given by

$$\left.\begin{aligned} E_x &= E_x',\ E_y = \gamma\left(E_y' + \frac{v}{c}H_z'\right),\ E_z = \gamma\left(E_z' - \frac{v}{c}H_y'\right);\\ H_x &= H_x',\ H_y = \gamma\left(H_y' - \frac{v}{c}E_z'\right),\ H_z = \gamma\left(H_z' + \frac{v}{c}E_y'\right). \end{aligned}\right\} \quad (21)$$

11·1 (4). *Relativistic Dynamics of a Charged Particle in an Electromagnetic Field.*

Suppose that the frame K is so chosen that the x-axis is parallel to the instantaneous velocity v of the charge e. Then in the frame K′ the charge is instantaneously at rest. We shall therefore assume that the usual electrostatic laws hold at that instant in K′ as far as the particle itself is concerned, so that the mechanical force acting upon it is

$$\vec{F}' = e\vec{E}'. \quad (1)$$

In this case it is found that the components of the force $\vec{F}'$ are transformed according to the law:

$$F_x = F_x',\ F_y = \frac{1}{\gamma} F_y',\ F_z = \frac{1}{\gamma} F_z'.$$

Then using (20) § 11·1 (3), we find from (1):

$$F_x = F_x' = eE_x' = eE_x,$$

$$F_y = \frac{1}{\gamma} F_y' = \frac{1}{\gamma} eE_y' = \frac{1}{\gamma} e\cdot\gamma \left(E_y - \frac{v}{c} H_z\right),$$

$$F_z = \frac{1}{\gamma} F_z' = \frac{1}{\gamma} eE_z' = \frac{1}{\gamma} e\cdot\gamma \left(E_z + \frac{v}{c} H_y\right). \tag{2}$$

Since the velocity $\vec{u}$ of the particle has components $(v, 0, 0)$ as observed by S at the instant, the equations (2) can be written in the vector form

$$\vec{F} = e\left\{\vec{E} + \frac{1}{c}(\vec{u} \times \vec{H})\right\}, \tag{3}$$

where $\vec{u} \times \vec{H}$ denotes the vector product.

The equations of motion are given by $\frac{d\vec{p}}{dt} = \vec{F}$, so that

$$\frac{d}{dt}\left(\frac{m_0\vec{u}}{\sqrt{1 - \frac{u^2}{c^2}}}\right) = e\left\{\vec{E} + \frac{1}{c}(\vec{u} \times \vec{H})\right\}$$

$$= e\left\{\left(-\operatorname{grad}\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right) + \frac{1}{c}\left(\vec{u} \times \operatorname{curl}\vec{A}\right)\right\}, \tag{4}$$

where m_0 is the rest-mass of the particle. The function T^* is defined as in (6) § 2·4 by

$$T^* = m_0 c^2 \{(1 - \sqrt{1 - \beta^2}\},\ \beta = \frac{u}{c}. \tag{5}$$

Then the x-component of the equation of motion (4) is

$$\frac{d}{dt}\left(\frac{\partial T^*}{\partial \dot{x}}\right) = -e\frac{\partial\phi}{\partial x} - \frac{e}{c}\frac{\partial A_x}{\partial t} + \frac{e}{c}(\vec{u} \times \operatorname{curl}\vec{A})_x. \tag{6}$$

But

$$(\vec{u} \times \operatorname{curl}\vec{A})_x$$

$$= \dot{y}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - \dot{z}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)$$

$$= \left(\dot{x}\,\frac{\partial A_x}{\partial x} + \dot{y}\,\frac{\partial A_y}{\partial y} + \dot{z}\,\frac{\partial A_z}{\partial x}\right) - \left(\dot{x}\,\frac{\partial A_x}{\partial x} + y\,\frac{\partial A_x}{\partial y} + \dot{z}\,\frac{\partial A_x}{\partial z}\right)$$

$$= \frac{\partial}{\partial x}\left(\dot{x}\,A_x + \dot{y}\,A_y + \dot{z}\,A_z\right) - \frac{dA_x}{dt} + \frac{\partial A_x}{\partial t}.$$

Hence (6) may be written as

$$\frac{d}{dt}\left(\frac{\partial T^*}{\partial x} + \frac{e}{c}\,A_x\right) - \frac{\partial}{\partial x}\left\{-\,e\phi + \frac{e}{c}\,(\vec{u}\cdot\vec{A})\right\} = 0. \tag{7}$$

But if we consider the co-ordinates of position and velocities, *viz.*, $x, y, z, \dot{x}, \dot{y}, \dot{z}$, then T^* does not depend on x, y, z explicitly, and ϕ does not depend on $x, \dot{y}, \dot{z}$, so that

$$\frac{\partial\phi}{\partial\dot{x}} = \frac{\partial\phi}{\partial\dot{y}} = \frac{\partial\phi}{\partial\dot{z}} = \frac{\partial T^*}{\partial x} = \frac{\partial T^*}{\partial y} = \frac{\partial T^*}{\partial z} = 0.$$

Also

$$A_x = \frac{\partial}{\partial\dot{x}}\left(\dot{x}\,A_x + \dot{y}\,A_y + \dot{z}\,A_z\right) = \frac{\partial}{\partial\dot{x}}\,(\vec{u}\cdot\vec{A}).$$

The equation of motion (7) can therefore be written as

$$\frac{d}{dt}\left\{\frac{\partial}{\partial\dot{x}}\left(T^* - e\phi + e\,\frac{\vec{u}\cdot\vec{A}}{c}\right)\right\} - \frac{\partial}{\partial x}\left\{T^* - e\phi + e\,\frac{\vec{u}\cdot\vec{A}}{c}\right\} = 0. \tag{8}$$

There are two similar equations for y and z. These are of the Lagrangian form with the Lagrangian function given by

$$L = T^* - e\phi + \frac{e}{c}\,(\vec{u}\cdot\vec{A}). \tag{9}$$

The momenta p_x, p_y, p_z are then

$$p_x = \frac{\partial L}{\partial\dot{x}} = \frac{\partial T^*}{\partial\dot{x}} + \frac{e}{c}\,A_x, \quad p_y = \frac{\partial T^*}{\partial\dot{y}} + \frac{e}{c}\,A_y, \quad p_z = \frac{\partial T^*}{\partial\dot{z}} + \frac{e}{c}\,A_z. \tag{10}$$

The Hamiltonian function H is then found to be

$$\begin{aligned} H &= \dot{x}\,p_x + \dot{y}p_y + \dot{z}\,p_z - L \\ &= \dot{x}\left(\frac{\partial T^*}{\partial\dot{x}} + \frac{e}{c}\,A_x\right) + \dot{y}\left(\frac{\partial T^*}{\partial\dot{y}} + \frac{e}{c}\,A_y\right) + \dot{z}\left(\frac{\partial T^*}{\partial\dot{z}} + \frac{e}{c}\,A_z\right) \\ &\quad - \left\{T^* - e\phi + \frac{e}{c}\,(\dot{x}\,A_x + \dot{y}\,A_y + \dot{z}\,A_z)\right\} \\ &= \dot{x}\,\frac{\partial T^*}{\partial\dot{x}} + \dot{y}\,\frac{\partial T^*}{\partial\dot{y}} + \dot{z}\,\frac{\partial T^*}{\partial\dot{z}} - T^* + e\phi. \end{aligned}$$

It was shown in (10) § 2·4 that

$$\sum_{xyz}\,\dot{x}\,\frac{\partial T^*}{\partial\dot{x}} - T^* = m_0c^2\left\{\frac{1}{\sqrt{1-\beta^2}} - 1\right\},$$

therefore we have

$$\mathrm{H} = m_0c^2\left\{\frac{1}{\sqrt{1-\beta^2}} - 1\right\} + e\phi. \tag{11}$$

To obtain the canonical form we require the expression of H in terms of p_x, p_y, p_z. For this, we see that

$$p_x - \frac{e}{c}\,\mathrm{A}_x = \frac{\partial \mathrm{T}^*}{\partial \dot{x}} = \frac{m_0\dot{x}}{\sqrt{1-\beta^2}}, \quad p_y - \frac{e}{c}\,\mathrm{A}_y = \frac{m_0\dot{y}}{\sqrt{1-\beta^2}},$$

$$p_z - \frac{e}{c}\,\mathrm{A}_z = \frac{m_0\dot{z}}{\sqrt{1-\beta^2}}.$$

Therefore

$$\Sigma\left(p_x - \frac{e}{c}\,\mathrm{A}_x\right)^2 = \frac{m_0{}^2u^2}{1-\dfrac{u^2}{c^2}},$$

so that

$$\Sigma\left(p_x - \frac{e}{c}\,\mathrm{A}_x\right)^2 + m_0{}^2c^2 = m_0{}^2\left(\frac{u^2}{1-\dfrac{u^2}{c^2}} + c^2\right) = \frac{m_0{}^2c^2}{1-\beta^2}.$$

Thus

$$\frac{m_0c^2}{\sqrt{1-\beta^2}} = c\left\{\Sigma\left(p_x - \frac{e}{c}\,\mathrm{A}_x\right)^2 + m_0{}^2c^2\right\}^{\frac{1}{2}}. \tag{12}$$

From (11) and (12) we obtain therefore

$$\mathrm{H} + m_0c^2 - e\phi = c\left\{\sum_{xyz}\left(p_x - \frac{e}{c}\,\mathrm{A}_x\right)^2 + m_0{}^2c^2\right\}^{\frac{1}{2}}. \tag{13}$$

If the particle were an electron, we would have to replace e by $-e$ throughout this section. We write also $\mathrm{E} = \mathrm{H} + m_0c^2$, E being the total energy, then from (13) we get for an electron

$$\mathrm{E} + e\phi = c\left\{\Sigma\left(p_x + \frac{e}{c}\,\mathrm{A}_x\right)^2 + m_0{}^2c^2\right\}^{\frac{1}{2}}, \tag{14}$$

or squaring,

$$\left(\frac{\mathrm{E}}{c} + \frac{e}{c}\,\phi\right)^2 = \sum_{xyz}\left(p_x + \frac{e}{c}\,\mathrm{A}_x\right)^2 + m_0{}^2c^2. \tag{15}$$

If there is no external field, we write $\phi = \mathrm{A}_x = \mathrm{A}_y = \mathrm{A}_z = 0$ in (13), and get the same Hamiltonian as (11) § 2·4.

If the velocity u of the electron can be taken to be very small compared to c, as assumed in the classical theory, then the term $m_0{}^2c^2$ under the root in (13) is very much larger than the rest.

Therefore, for the classical Hamiltonian H′ we get

$$H' + m_0c^2 - e\phi = m_0c^2\left\{1 + \frac{1}{m_0^2c^2}\Sigma\left(p_x - \frac{e}{c}A_x\right)^2\right\}^{\frac{1}{2}}$$
$$= m_0c^2\left\{1 + \frac{1}{2m_0^2c^2}\Sigma\left(p_x - \frac{e}{c}A_x\right)^2\right\},$$

neglecting second and higher powers. Thus

$$H' = e\phi + \frac{1}{2m_0}\sum_{xyz}\left(p_x - \frac{e}{c}A_x\right)^2. \tag{16}$$

For the electron, we have therefore

$$H' = -e\phi + \frac{1}{2m_0}\sum_{xyz}\left(p_x + \frac{e}{c}A_x\right)^2. \tag{17}$$

11·2. *Earlier Attempts at a Relativistic Quantum Theory.*

The quantum mechanics developed so far has taken no account of the theory of relativity. This deficiency was supplied in the middle of the year 1926 by Schrödinger himself, and later by many other workers. Their contribution consisted in generalising the wave-equation to a relativistic form.

Thus, for a particle of mass m, Cartesian co-ordinates x, y, z and corresponding momenta p_x, p_y, p_z, moving under a conservative field of force of potential V, we had the classical Hamiltonian

$$H = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + V. \tag{1}$$

From this Hamiltonian, we derived the wave-equation by taking the momenta p_x, p_y, p_z, as operators defined by

$$p_x = \frac{h}{2\pi i}\frac{\partial}{\partial x}, \quad p_y = \frac{h}{2\pi i}\frac{\partial}{\partial y}, \quad p_z = \frac{h}{2\pi i}\frac{\partial}{\partial z}, \tag{2}$$

so that the wave-equation became

$$(H - E) \quad = 0, \tag{3}$$

where E is defined as the operator

$$E = -\frac{h}{2\pi i}\frac{\partial}{\partial t}. \tag{4}$$

To generalise the equation (3) for the relativity effect, we shall assume first that the particle of rest mass m_0 is moving in a free space, *i.e.*, V = 0. The velocity u of the particle is given by $u^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$. If we write $\beta = \frac{u}{c}$, the energy E is given by

$$E = \frac{m_0c^2}{\sqrt{1-\beta^2}}. \tag{5}$$

Further, for the momenta we have

$$p_x = \frac{m_0\dot{x}}{\sqrt{1-\beta^2}}, \quad p_y = \frac{m_0\dot{y}}{\sqrt{1-\beta^2}}, \quad p_z = \frac{m_0\dot{z}}{\sqrt{1-\beta^2}};$$

hence

$$p_x^2 + p_y^2 + p_z^2 = \frac{m_0^2u^2}{1-\beta^2} = \frac{m_0^2c^2\beta^2}{1-\beta^2} = m_0^2c^2\left(\frac{1}{1-\beta^2} - 1\right),$$

which, on account of (5) gives

$$p_x^2 + p_y^2 + p_z^2 = \frac{E^2}{c^2} - m_0^2c^2,$$

or

$$p_x^2 + p_y^2 + p_z^2 + m_0 c^2 - \frac{E^2}{c^2} = 0.^* \qquad (6)$$

The relativistic wave-equation would therefore be

$$\left(p_x^2 + p_y^2 + p_z^2 + m_0^2 c^2 - \frac{E^2}{c^2}\right)\psi = 0, \qquad (7)$$

or on substituting from (2) and (4)

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{4\pi^2}{h^2}\ m_0^2c^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\psi = 0. \qquad (8)$$

We write $p_t = \frac{E}{c}$ and the wave-equation (7) for a free electron can be written as

$$(-p_t^2 + p_x^2 + p_y^2 + p_z^2 + m_0^2 c^2)\psi = 0. \qquad (9)$$

Now consider an external electromagnetic field of scalar potential A_t, and vector potential $\vec{A}$ with components A_x, A_y, A_z, and let the moving particle be an electron of charge $-e$. Then if we write $p_t = \frac{E}{c}$, the equation (15) of § 11·1 (4) becomes

$$-\left(p_t + \frac{e}{c}A_t\right)^2 + \left(p_x + \frac{e}{c}A_x\right)^2 + \left(p_y + \frac{e}{c}A_y\right)^2$$
$$+\left(p_z + \frac{e}{c}A_z\right)^2 + m_0^2 c^2 = 0,$$

or, on setting

$$g_t = p_t + \frac{e}{c}A_t, \quad g_x = p_x + \frac{e}{c}A_x, \quad g_y = p_y + \frac{e}{c}A_y, \quad g_z = p_z + \frac{e}{c}A_z, \qquad (10)$$

it becomes:

$$-g_t^2 + g_x^2 + g_y^2 + g_z^2 + m_0^2 c^2 = 0. \qquad (11)$$

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* This could be obtained immediately from (15) § 11·1 (4) by setting $\phi = A_x = A_y = A_z = 0$.

This means that in the case of an external electromagnetic field, the p's have to be replaced by the g's. This, if we write

$$\mathrm{F} \equiv -g_t^2 + g_x^2 + g_y^2 + g_z^2 + m_0^2 c^2,$$

the relativistic wave-equation becomes

$$\mathrm{F}\psi \equiv (-g_t^2 + g_x^2 + g_y^2 + g_z^2 + m_0^2 c^2)\,\psi = 0, \tag{12}$$

where ψ is a function of x, y, z, t.

When the relativistic wave-equations (9) and (12) were applied to atomic problems, it was found that the number of stationary states obtained from these equations was half the number required by experiments. Pauli then pointed out that to remove this deficiency, one must introduce the idea of the spinning electron. According to this hypothesis an electron possesses a spin angular momentum $\frac{1}{2}\frac{h}{2\pi}$, and a magnetic moment $\frac{eh}{4m_0 c}$. Pauli and Darwin fitted this hypothesis into the relativistic wave-equation, and showed that the above-mentioned difficulty of the wrong number of stationary states disappeared if account was taken of the electron-spin.

Early in 1928 Dirac pointed out that the spin theory was open to several objections. At the very outset, it is obvious that the idea of a spinning electron is a completely arbitrary assumption. In Dirac's own words, it remains incomprehensible " why Nature should have chosen this particular model for the electron instead of being satisfied with the point charge ". Then again, on the spinning electron model, we should expect that if the electron is moving in a central field of force, the magnitude of its resultant orbital angular momentum should be constant. But, it turns out that this magnitude is not constant, and the model therefore fails in this respect.

Another, and a very serious, objection to the theory of the present section is that the relativistic wave-equations (9) and (12) are non-linear in E or $\frac{\partial}{\partial t}$. The non-relativistic wave-equation (3) is linear in $\frac{\partial}{\partial t}$, so that if the wave-function ψ_0 is known at time $t = 0$, its value at any time t is determined in terms of ψ_0. This property does not hold for the equations (9) and (12) on account of their being non-linear in $\frac{\partial}{\partial t}$.

Finally, it should be remarked that Pauli's spin theory was successful only for those electrons whose velocities were not too great.

A reformulation of the relativistic wave-theory was given by Dirac early in 1928. He showed that if the wave-equation is set up so as to satisfy the requirements of the principles of relativity and quantum mechanics, the existence of the angular momentum and magnetic moment of the electron can be deduced without any arbitrary assumptions about the spin of the electron.

11·3. *Dirac's Relativistic Theory for a Free Electron.*

Dirac looked for a wave-equation which should be invariant under a Lorentz transformation, and should be equivalent to the non-relativistic equation (8) § 11·2. As pointed out in the last section, this equation should be linear in $\partial/\partial t$, *i.e.*, in p_t, and therefore on account of symmetry, it should be linear in p_x, p_y, p_z also.

Dirac assumed therefore the relativistic wave-equation to be of the form

$$(p_t + \alpha_1 p_x + \alpha_2 p_y + \alpha_3 p_z + \alpha_4 m_0 c)\,\psi = 0, \qquad (1)$$

where α_1, α_2, α_3, α_4 are dynamical variables or operators which are independent of the p's, thus commuting with t, x, y, z. Moreover, for a particle moving in empty space, all the points in space are equivalent, and therefore the Hamiltonian cannot involve t, x, y, z. The α's are therefore independent of t, x, y, z, and consequently commute with the p's. This means that α_1, α_2, α_3, α_4 are functions of other dynamical variables and not of the co-ordinates and momenta of the electron, and that the wave-function ψ is a function also of these other variables besides t, x, y, z. These other variables are the so-called " spin matrices " or " spinors," which we shall now investigate.

Multiplying (1) by the operator $(-p_t + \alpha_1 p_x + \alpha_2 p_y + \alpha_3 p_z + \alpha_4 m_0 c)$ on the left, we get

$$(-p_t + \alpha_1 p_x + \alpha_2 p_y + \alpha_3 p_z + \alpha_4 m_0 c)\,(p_t + \alpha_1 p_x + \alpha_2 p_y + \alpha_3 p_z + \alpha_4 m_0 c)\,\psi = 0,$$

or

$$\{-p_t^2 + (\alpha_1^2 p_x^2 + \alpha_2^2 p_y^2 + \alpha_3^2 p_z^2) + (\alpha_1\alpha_2 + \alpha_2\alpha_1)\,p_x p_y + (\alpha_2\alpha_3 + \alpha_3\alpha_2)\,p_y p_z + (\alpha_3\alpha_1 + \alpha_1\alpha_3)\,p_z p_x + \alpha_4^2 m_0^2 c^2 + (\alpha_1\alpha_4 + \alpha_4\alpha_1)\,m_0 c p_x + (\alpha_2\alpha_4 + \alpha_4\alpha_2)\,m_0 c p_y + (\alpha_3\alpha_4 + \alpha_4\alpha_3)\,m_0 c p_z\}\,\psi = 0. \qquad (2)$$

This equation agrees with the equation (9) § 11·2 if the α's satisfy the following relations

$$\alpha_\mu{}^2 = 1, \qquad \alpha_\mu \alpha_\nu + \alpha_\nu \alpha_\mu = 0 \quad (\mu \neq \nu),$$
$$(\mu, \nu = 1, 2, 3, 4). \tag{3}$$

The problem now reduces to that of finding four dynamical variables α_μ which satisfy the relations (3). In quantum mechanics, these dynamical variables are matrices. Dirac obtains them with the help of six other matrices σ_1, σ_2, σ_3, ρ_1, ρ_2, ρ_3, which are of the fourth order, and which are as follows :—

$$\sigma_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix},$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \rho_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$\rho_2 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \qquad \rho_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

It can be easily verified that the σ's and ρ's satisfy the following relations for all $r, s, t, = 1, 2, 3$:—

$$\sigma_r{}^2 = 1, \quad \sigma_r\, \sigma_s + \sigma_s\, \sigma_r = 0, \quad (r \neq s), \tag{4}$$
$$\rho_r{}^2 = 1, \quad \rho_r\, \rho_s + \rho_s\, \rho_r = 0, \quad (r \neq s), \tag{5}$$
$$\rho_r\, \sigma_t = \sigma_t\, \rho_r. \tag{6}$$

The α's are now defined by the following equations* :—

$$\alpha_1 = \rho_1\sigma_1, \quad \alpha_2 = \rho_1\sigma_2, \quad \alpha_3 = \rho_1\sigma_3, \quad \alpha_4 = \rho_3. \tag{7}$$

Obviously, the α's so defined satisfy the relations (3) in virtue of the properties (4), (5) and (6).

The wave-equation (1) now takes the form :

$$\{p_t + \rho_1 (\sigma_1 p_x + \sigma_2 p_y + \sigma_3 p_z) + \rho_3 m_0 c\}\, \psi = 0. \tag{8}$$

This is Dirac's wave-equation for a free electron.

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* The matrices for the α's are given in § 11·7.

11 · 4. *The Electron under an Arbitrary Field.*

Now suppose that the electron is moving in an external electromagnetic field of scalar potential A_t and vector potential $\vec{A}$. We have shown in § 11·2 that in order to get the Hamiltonian for this case, we must substitute $p_t + \frac{e}{c} A_t$ for p_t, $p_x + \frac{e}{c} A_x$, for p_x, etc., in the Hamiltonian for the case of a free electron. Carrying out this substitution, and remembering the definitions of the g's [(10) § 11 · 2], we get from (8) § 11 · 3 :

$$\{g_t + \rho_1 (\sigma_1 g_x + \sigma_2 g_y + \sigma_3 g_z) + \rho_3 m_0 c\} \psi = 0. \tag{1}$$

Writing down the values of the operators g_t, g_x, g_y, g_z, we get Dirac's relativistic wave-equation :

$$\left[\left(-\frac{h}{2\pi ic}\frac{\partial}{\partial t} + \frac{e}{c} A_t\right) + \rho_1 \left\{\sigma_1 \frac{h}{2\pi i}\frac{\partial}{\partial_x} + \frac{e}{c} A_x\right) + \right.$$
$$\left. + \sigma_2 \left(\frac{h}{2\pi i}\frac{\partial}{\partial y} + \frac{e}{c} A_y\right) + \sigma_3 \left(\frac{h}{2\pi i}\frac{\partial}{\partial z} + \frac{e}{c} A_z\right)\right\} + \rho_3 m_0 c\Big]\psi = 0, \tag{2}$$

or

$$\left\{\left(-\frac{h}{2\pi ic}\frac{\partial}{\partial t} + \frac{e}{c} A_t\right) + \alpha_1 \left(\frac{h}{2\pi i}\frac{\partial}{\partial x} + \frac{e}{c} A_x\right) + \alpha_2 \left(\frac{h}{2\pi i}\frac{\partial}{\partial y} + \frac{e}{c} A_y\right)\right.$$
$$\left. + \alpha_3 \left(\frac{h}{2\pi i}\frac{\partial}{\partial y} + \frac{e}{c} A_z\right) + \alpha_4 m_0 c\right\} \psi = 0. \tag{3}$$

For the purpose of this section only, suppose that we take a representation of the α's in which all the elements of the matrices representing α_1, α_2, α_3 are real, and all the elements of the matrix representing α_4 are pure imaginary. This is, for instance, possible if in (7) § 11 · 3 we take $\alpha_2 = \rho_3$ and $\alpha_4 = \rho_1\sigma_2$. Then, if we change the sign of i in (3), we see that the complex conjugate of (3) also holds, *viz.*,

$$\left\{\left(\frac{h}{2\pi ic}\frac{\partial}{\partial t} + \frac{e}{c} A_t\right) + \alpha_1 \left(-\frac{h}{2\pi i}\frac{\partial}{\partial x} + \frac{e}{c} A_x\right) + \alpha_2 \left(-\frac{h}{2\pi i}\frac{\partial}{\partial y} + \frac{e}{c} A_y\right)\right.$$
$$\left. + \alpha_3 \left(-\frac{h}{2\pi i}\frac{\partial}{\partial z} + \frac{e}{c} A_z\right) - \alpha_4 m_0 c\right\} \psi = 0. \tag{4}$$

Now, if we had an electron with a positive charge $+e$, the Equation (4) would become :

$$\left\{\left(\frac{h}{2\pi ic}\frac{\partial}{\partial t} - \frac{e}{c} A_t\right) - \alpha_1 \left(\frac{h}{2\pi i}\frac{\partial}{\partial x} + \frac{e}{c} A_x\right) - \alpha_2 \left(\frac{h}{2\pi i}\frac{\partial}{\partial y} + \frac{e}{c} A_y\right)\right.$$
$$\left. - \alpha_3 \left(\frac{h}{2\pi i}\frac{\partial}{\partial z} + \frac{e}{c} A_z\right) - \alpha_4 m_0 c\right\} \psi = 0. \tag{5}$$

This is the same equation as (3). It is obtained from (3) by multiplying both sides by -1. Thus we see that the wave-equation (2) refers to a negative electron (with charge $-e$) as well as to a "positive electron" (with charge $+e$). Half the solution of (2) would refer to the negative electron, and half to the positive electron.

On account of the matrices α being of the fourth order, the wave-function ψ will have four components, two of which will correspond to the negative electron having states of positive energy, and the other two would correspond to the positive electron having states of negative energy. At the time when Dirac first published his theory, the postive electron, or the "positron", was not known to exist. It is generally recognised that with his relativistic electron theory Dirac had predicted the existence of these particles. As remarked in § 3 · 4, the existence of the positron was experimentally demonstrated in 1932 by Carl Anderson.

Dirac has also proved that his relativistic wave-equations (8) § 11 · 3 and (1) § 11 · 4 are invariant under a Lorentz-transformation.

11·5 *Existence of the Magnetic Moment of the Electron.*

To find out how Dirac's wave-equation (1) § 11·4 differs from the non-relativistic wave-equation (10) § 11·2, we multiply the former by a factor, just as we did in § 11·3 to get equation (2) from (1). Remembering the definitions of the g's, *viz.*,

$$g_t = p_t + \frac{e}{c}A_t,\ g_x = p_x + \frac{e}{c}A_x,\ g_y = p_y + \frac{e}{c}A_y,\ g_z = p_z + \frac{e}{c}A_z, \tag{1}$$

Dirac's equation is written in the form

$$\{g_t + \rho_1(\sigma_1 g_x + \sigma_2 g_y + \sigma_3 g_z) + \rho_3 m_0 c\}\psi = 0. \tag{2}$$

Multiplying (2) on the left with the factor

$$-g_t + \rho_1(\sigma_1 g_x + \sigma_2 g_y + \sigma_3 g_z) + \rho_3 m_0 c,$$

we obtain

$$\{-g_t + \rho_1(\sigma_1 g_x + \sigma_2 g_y + \sigma_3 g_z) + \rho_3 m_0 c\}\{g_t + \rho_1(\sigma_1 g_x + \sigma_2 g_y + \sigma_3 g_z) + \rho_3 m_0 c\}\psi = 0,$$

or, on account of (4), (5), (6), § 11·3,

$$\{-g_t^2 + (\sigma_1 g_x + \sigma_2 g_y + \sigma_3 g_z)^2 + m_0^2c^2 + \rho_1(\sigma_1 g_x g_t - g_t\sigma_1 g_x + \sigma_2 g_y g_t - g_t\sigma_2 g_y + \sigma_3 g_z g_t - g_t\sigma_3 g_z)\}\psi = 0,$$

or, since the σ's commute with the g's:

$$\{-g_t^2 + (\sigma_1 g_x + \sigma_2 g_y + \sigma_3 g_z)^2 + m_0^2c^2 + \rho_1 [\sigma_1 (g_x g_t - g_t g_x) + \sigma_2 (g_y g_t - g_t g_y) + \sigma_3 (g_z g_t - g_t g_z)]\} \psi = 0. \tag{3}$$

Now

$$(\sigma_1 g_x + \sigma_2 g_y + \sigma_3 g_z)^2 = (\sigma_1 g_x + \sigma_2 g_y + \sigma_3 g_z)(\sigma_1 g_x + \sigma_2 g_y + \sigma_3 g_z)$$
$$= (\sigma_1^2 g_x^2 + \sigma_2^2 g_y^2 + \sigma_3^2 g_z^2) + [(\sigma_1\sigma_2 g_x g_y + \sigma_2\sigma_1 g_y g_x) + (\sigma_2\sigma_3 g_y g_z + \sigma_3\sigma_2 g_z g_y) + (\sigma_3\sigma_1 g_z g_x + \sigma_1\sigma_3 g_x g_z)].$$

But from the definitions of the σ matrices, it can be easily proved that

$$\sigma_1\sigma_2 = i\sigma_3 = -\sigma_2\sigma_1, \quad \sigma_2\sigma_3 = i\sigma_1 = -\sigma_3\sigma_2, \quad \sigma_3\sigma_1 = i\sigma_2 = -\sigma_1\sigma_3,$$

and as shown in (4) § 11·3 (4)

$$\sigma_1^2 = 1, \quad \sigma_2^2 = 1, \quad \sigma_3^2 = 1.$$

We have therefore

$$(\sigma_1 g_x + \sigma_2 g_y + \sigma_3 g_z)^2 = (g_x^2 + g_y^2 + g_z^2) + i\{\sigma_3 (g_x g_y - g_y g_x) + \sigma_1 (g_y g_z - g_z g_y) + \sigma_2 (g_z g_x - g_x g_z)\}. \tag{5}$$

Now consider

$$\left(g_x g_y - g_y g_x\right)\psi = \left\{\left(p_x + \frac{e}{c}A_x\right)\left(p_y + \frac{e}{c}A_y\right) - \left(p_y + \frac{e}{c}A_y\right)\left(p_x + \frac{e}{c}A_x\right)\right\}\psi$$

$$= \left\{\left(\frac{h}{2\pi i}\frac{\partial}{\partial x} + \frac{e}{c}A_x\right)\left(\frac{h}{2\pi i}\frac{\partial}{\partial y} + \frac{e}{c}A_y\right) - \left(\frac{h}{2\pi i}\frac{\partial}{\partial y} + \frac{e}{c}A_y\right)\left(\frac{h}{2\pi i}\frac{\partial}{\partial x} + \frac{e}{c}A_x\right)\right\}\psi$$

$$= \left(\frac{h}{2\pi i}\right)^2 \frac{\partial^2\psi}{\partial x\partial y} + \frac{h}{2\pi i}\frac{\partial}{\partial x}\left(\frac{e}{c}A_y\psi\right) - \frac{e}{c}A_x\frac{h}{2\pi i}\frac{\partial\psi}{\partial y} + \frac{e^2}{c^2}A_x A_y\psi$$
$$- \left(\frac{h}{2\pi i}\right)^2 \frac{\partial^2\psi}{\partial y\partial x} - \frac{h}{2\pi i}\frac{\partial}{\partial y}\left(\frac{e}{c}A_x\psi\right) - \frac{e}{c}A_y\frac{h}{2\pi i}\frac{\partial\psi}{\partial x} - \frac{e^2}{c^2}A_y A_x\psi$$

$$= \frac{he}{2\pi ic}\frac{\partial}{\partial x}(A_y\psi) + \frac{he}{2\pi ic}A_x\frac{\partial x}{\partial y} - \frac{he}{2\pi ic}\frac{\partial}{\partial y}(A_x\psi) - \frac{he}{2\pi ic}A_y\frac{\partial\psi}{\partial x}$$
$$= \frac{he}{2\pi ic}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\psi$$

$$= \frac{he}{2\pi ic}H_z\psi, \tag{6}$$

because of (1) § 11·1.

Similarly we get

$$(g_y g_z - g_z g_y)\,\psi = \frac{he}{2\pi ic}\left(\frac{\partial \mathrm{A}_z}{\partial y} - \frac{\partial \mathrm{A}_y}{\partial z}\right)\psi = \frac{he}{2\pi ic}\,\mathrm{H}_x\,\psi, \tag{6 a}$$

$$(g_z g_x - g_x g_z)\,\psi = \frac{he}{2\pi ic}\left(\frac{\partial \mathrm{A}_x}{\partial z} - \frac{\partial \mathrm{A}_z}{\partial x}\right)\psi = \frac{he}{2\pi ic}\,\mathrm{H}_y\,\psi. \tag{6 b}$$

Further, we find

$$(g_x g_t - g_t g_x)\,\psi = \left\{\left(p_x + \frac{e}{c}\,\mathrm{A}_x\right)\left(p_t + \frac{e}{c}\,\mathrm{A}_t\right) - \left(p_t + \frac{e}{c}\,\mathrm{A}_t\right)\left(p_x + \frac{e}{c}\,\mathrm{A}_x\right)\right\}\psi,$$

$$= \left\{\left(\frac{h}{2\pi i}\frac{\partial}{\partial x} + \frac{e}{c}\,\mathrm{A}_x\right)\left(\frac{-h}{2\pi ic}\frac{\partial}{\partial t} + \frac{e}{c}\,\mathrm{A}_t\right) - \left(\frac{-h}{2\pi ic}\frac{\partial}{\partial t} + \frac{e}{c}\,\mathrm{A}_t\right)\left(\frac{h}{2\pi i}\frac{\partial}{\partial x} + \frac{e}{c}\,\mathrm{A}_x\right)\right\}\psi,$$

$$= -\left(\frac{h}{2\pi i}\right)^2\frac{1}{c}\frac{\partial^2\psi}{\partial x\partial t} + \frac{h}{2\pi i}\frac{\partial}{\partial x}\left(\frac{e}{c}\,\mathrm{A}_t\,\psi\right) - \frac{e}{c^2}\,\mathrm{A}_x\,\frac{h}{2\pi i}\frac{\partial\psi}{\partial t} + \frac{e^2}{c^2}\,\mathrm{A}_x\mathrm{A}_t\psi$$

$$+ \left(\frac{h}{2\pi i}\right)^2\frac{1}{c}\frac{\partial^2\psi}{\partial x\partial t} + \frac{h}{2\pi ic}\frac{\partial}{\partial t}\left(\frac{e}{c}\,\mathrm{A}_x\,\psi\right) - \frac{e}{c}\,\mathrm{A}_t\,\frac{h}{2\pi i}\frac{\partial\psi}{\partial x} - \frac{e^2}{c^2}\,\mathrm{A}_t\mathrm{A}_x\psi,$$

$$= \frac{he}{2\pi ic}\frac{\partial}{\partial x}(\mathrm{A}_t\,\psi) - \frac{he}{2\pi ic^2}\,\mathrm{A}_x\frac{\partial\psi}{\partial t} + \frac{he}{2\pi ic^2}\frac{\partial}{\partial t}(\mathrm{A}_x\,\psi) - \frac{he}{2\pi ic}\,\mathrm{A}_t\,\frac{\partial\psi}{\partial x},$$

$$= \frac{he}{2\pi ic}\left(\frac{\partial \mathrm{A}_t}{\partial x} + \frac{1}{c}\frac{\partial \mathrm{A}_x}{\partial t}\right)\psi = \frac{-he}{2\pi ic}\,\mathrm{E}_x\,\psi, \tag{7}$$

because of (1) § 11·1

Similarly,

$$(g_y g_t - g_t g_y)\,\psi = \frac{he}{2\pi ic}\left(\frac{\partial \mathrm{A}_t}{\partial y} + \frac{1}{c}\frac{\partial \mathrm{A}_y}{\partial t}\right)\psi = \frac{-he}{2\pi ic}\,\mathrm{E}_y\,\psi, \tag{7 a}$$

$$(g_z g_t - g_t g_z)\,\psi = \frac{he}{2\pi ic}\left(\frac{\partial \mathrm{A}_t}{\partial z} + \frac{1}{c}\frac{\partial \mathrm{A}_z}{\partial t}\right)\psi = \frac{-he}{2\pi ic}\,\mathrm{E}_z\,\psi. \tag{7 b}$$

Substituting (5), (6), (7) in (3), we obtain

$$\Big\{-g_t^2 + g_1^2 + g_2^2 + g_3^2 + m_0^2c^2 + \frac{he}{2\pi c}(\sigma_1\,\mathrm{H}_x + \sigma_2\mathrm{H}_y + \sigma_3\mathrm{H}_z) + \frac{ihe}{2\pi c}\,\rho_1(\sigma_1\mathrm{E}_x + \sigma_2\mathrm{E}_y + \sigma_3\mathrm{E}_z)\Big\}\,\psi = 0 \tag{8}$$

or, substituting the value of the g's from (1) :

$$\Big\{-\Big(p_t+\frac{e}{c}A_t\Big)^2+\Big(p_x+\frac{e}{c}A_x\Big)^2+\Big(p_y+\frac{e}{c}A_y\Big)^2+$$
$$\Big(p_z+\frac{e}{c}A_z\Big)^2+m^2c^2+\frac{he}{2\pi c}(\sigma_1H_x+\sigma_2H_y+\sigma_3H_z)$$
$$+\frac{ihe}{2\pi c}\rho_1(\sigma_1E_x+\sigma_2E_y+\sigma_3E_z)\Big\}\psi=0. \tag{9}$$

The Hamiltonian in this equation differs from that in (10) § 11·3 by the terms $\frac{he}{2\pi c}(\sigma_1H_x+\sigma_2H_y+\sigma_3H_z)+\frac{ihe}{2\pi c}\rho_1(\sigma_1E_x+\sigma_2E_y+\sigma_3E_z)$. These two terms, divided by $2m_0$, can be considered as an increase in the potential energy of the electron due to its spin. Thus the additional potential energy is

$$\frac{1}{2m_0}\cdot\frac{he}{2\pi c}(\sigma_1H_x+\sigma_2H_y+\sigma_3H_z)$$
$$+\frac{1}{2m_0}\frac{he}{2\pi c}i\rho_1(\sigma_1E_x+\sigma_2E_y+\sigma_3E_z). \tag{10}$$

The electron will therefore behave as though it has a magnetic moment with components

$$\Big(\frac{eh}{4\pi m_0c}\sigma_1,\ \frac{eh}{4\pi m_0c}\sigma_2,\ \frac{eh}{4\pi m_0c}\sigma_3\Big), \tag{11}$$

and an electric moment with components

$$\Big(\frac{eh}{4\pi m_0c}i\rho_1\sigma_1,\ \frac{eh}{4\pi m_0c}i\rho_1\sigma_2,\ \frac{eh}{4\pi m_0c}i\rho_1\sigma_3\Big). \tag{12}$$

The magnetic moment has, therefore, just the amount assumed in the spinning electron theory, § 11·2, and which is here derived, according to Dirac, as a natural consequence of the relativistic wave-equation.

The electric moment, however, is a pure imaginary quantity, and cannot be considered to have any physical meaning. It has arisen only after multiplying the real Hamiltonian in (1) § 11·5 with a factor in order to compare it with the classical Hamiltonian in (10) § 11·2. This being rather an artificial operation, we must not be surprised if it leads to some terms which have no physical meaning.

11·6. *Existence of the Spin Angular Momentum of the Electron.*

We remarked at the end of § 11·2 that the spinning electron theory postulated the existence of an intrinsic (spin) angular

momentum of the electron of amount $\frac{1}{2}\frac{h}{2\pi}$. We have now to prove the existence of this spin angular momentum on Dirac's theory. Since this angular momentum does not give rise to any potential energy, it did not appear in the result of calculations of § 11·5. To demonstrate its existence, we shall consider the special case of the motion of an electron in a central field of force, and shall determine the angular momentum integrals.

Take the centre of force as origin, rectangular Cartesian co-ordinates x, y, z and corresponding momenta p_x, p_y, p_z. μ is the mass of the electron, and V the potential energy which is a function of $x^2 + y^2 + z^2$.

The classical Hamiltonian, neglecting the relativity effect, is

$$\mathrm{H} = \frac{1}{2\mu}(p_x^2 + p_y^2 + p_z^2) + \mathrm{V}. \tag{1}$$

If $\vec{m}$ is the orbital angular momentum of the electron, we proved in § 6·8 (1) that each of m_x, m_y, m_z commutes with any function of $x^2 + y^2 + z^2$ and $p_x^2 + p_y^2 + p_z^2$. The Hamiltonian H, as given in (1), being a function of $x^2 + y^2 + z^2$ and $p_x^2 + p_y^2 + p_z^2$, we see that each of m_x, m_y, m_z, commutes with H. We conclude that the orbital angular momentum is a constant of the motion, just as in Newtonian dynamics.

However, if we take account of the relativity effect also, it turns out that the orbital angular momentum $\vec{m}$ alone is not a constant of the motion. To get such a constant, we have to add another term which comes out to be just the required spin angular momentum.

The force on the electron is given by the potential V (r) which can be taken to be the scalar potential A_t, the vector potential $\vec{\mathrm{A}} \equiv (\mathrm{A}_x, \mathrm{A}_y, \mathrm{A}_z)$ being zero. Thus

$$\mathrm{A}_x = \mathrm{A}_y = \mathrm{A}_z = 0, \quad \mathrm{A}_t = \mathrm{V}(r). \tag{1}$$

From Dirac's wave-equation (1) § 11·4, we get then

$$\mathrm{F}\psi \equiv \left\{p_t + \frac{e}{c}\mathrm{V} + \rho_1(\sigma_1 p_x + \sigma_2 p_y + \sigma_3 p_z) + \rho_3 m_0 c\right\}\psi = 0. \tag{2}$$

To get the Hamiltonian H, we remark that $p_t = \frac{\mathrm{E}}{c}$, where E is the energy parameter. (Since we require only periodic solutions

of the wave-equation $F\psi = 0$, p_t is no more considered as an operator).

The equation (2) can therefore be written as

$$\left\{\frac{E}{c} - \left[-\frac{e}{c}V - \rho_1(\sigma_1 p_x + \sigma_2 p_y + \sigma_3 p_z) - \rho_3 m_0 c\right]\right\}\psi = 0,$$

or, on multiplying throughout by c,

$$\{E - [-eV - c\rho_1(\sigma_1 p_x + \sigma_2 p_y + \sigma_3 p_z) - \rho_3 m_0 c^2]\}\psi = 0. \tag{3}$$

Thus we see that the relativistic Hamiltonian is

$$H = -eV - c\rho_1(\sigma_1 p_x + \sigma_2 p_y + \sigma_3 p_z) - \rho_3 m_0 c^2. \tag{4}$$

The classical Hamiltonian would have been simply $-eV$, so that the last two terms in (4) are contributed by the relativity-effect.

We shall prove now that the orbital angular momentum $\vec{m}$ is not a constant of the motion, *i.e.*, the components m_x, m_y, m_z as defined in (1) § 6·8 (1) do not commute with H given by (4).

For

$$\begin{aligned} m_x H - H m_x &= -c\rho_1 \{m_x(\sigma_1 p_x + \sigma_2 p_y + \sigma_3 p_z) - (\sigma_1 p_x + \sigma_2 p_y \\ &\qquad + \sigma_3 p_z)m_x\} \\ &= -c\rho_1\{\sigma_1(m_x p_x - p_x m_x) + \sigma_2(m_x p_y - p_y m_x) \\ &\qquad + \sigma_3(m_x p_z - p_z m_x)\} \\ &= -\frac{ih}{2\pi}c\rho_1\{\sigma_2 p_z - \sigma_3 p_y\}\psi \neq 0, \end{aligned} \tag{5}$$

on account of (5) § 6·8 (1). It can be similarly proved that m_y and m_z do not commute with H. Thus the orbital angular momentum is not a constant of the motion.

We have further from (4)

$$\begin{aligned} \frac{1}{2}\frac{h}{2\pi}\sigma_1 H - H\frac{1}{2}\frac{h}{2\pi}\sigma_1 &= -\frac{hc}{4\pi}\rho_1\{\sigma_1(\sigma_1 p_x + \sigma_2 p_y + \sigma_3 p_z) \\ &\qquad - (\sigma_1 p_x + \sigma_2 p_y + \sigma_3 p_z)\sigma_1\} \\ &= -\frac{hc}{4\pi}\rho_1\left\{(\sigma_1\sigma_2 - \sigma_2\sigma_1)p_y + (\sigma_2\sigma_3 - \sigma_3\sigma_1)p_z\right\} \\ &= -\frac{hc}{4\pi}\rho_1\left\{2i\sigma_3 p_y - 2i\sigma_2 p_z\right\}, \end{aligned}$$

on account of (4) § 11·5. Thus we get

$$\frac{1}{2}\frac{h}{2\pi}\sigma_1 H - H\frac{1}{2}\frac{h}{2\pi}\sigma_1 = -\frac{ih}{2\pi}c\rho_1\left\{\sigma_3 p_y - \sigma_2 p_z\right\}. \tag{6}$$

From (5) and (6) we find therefore that

$$\left(m_x + \frac{1}{2}\frac{h}{2\pi}\sigma_1\right)\mathrm{H} - \mathrm{H}\left(m_x + \frac{1}{2}\frac{h}{2\pi}\sigma_1\right) = 0, \qquad (7)$$

and similarly, that $m_y + \frac{1}{2}\frac{h}{2\pi}\sigma_2$ and $m_z + \frac{1}{2}\frac{h}{2\pi}\sigma_3$ commute with H. Thus if we define $\vec{\mathrm{M}}$ by the relations

$$\mathrm{M}_x = m_x + \frac{1}{2}\frac{h}{2\pi}\sigma_1,\ \mathrm{M}_y = m_y + \frac{1}{2}\frac{h}{2\pi}\sigma_2,\ \mathrm{M}_z = m_z + \frac{1}{2}\frac{h}{2\pi}\sigma_3, \qquad (8)$$

we see that $\vec{\mathrm{M}}$, and not the orbital angular momentum $\vec{m}$ alone, is a constant of the motion.

This result can be interpreted to mean that the electron has a spin angular momentum with components $\left(\frac{1}{2}\frac{h}{2\pi}\sigma_1,\ \frac{1}{2}\frac{h}{2\pi}\sigma_3,\ \frac{1}{2}\frac{h}{2\pi}\sigma_3\right)$, which is just the spin angular momentum postulated in the spinning electron theory of § 11·2.

11·7. *The Theory of the Hydrogen Atom; Fine Structure of Spectral Lines.*

As a particular example of Dirac's theory, we shall solve the problem of the hydrogen-atom, obtaining the fine structure of the lines observed in instruments of high resolving power. Sommerfeld's treatment of this problem on Bohr's theory has been described in § 5·7. Dirac solved the problem as an example of his theory, but immediately afterwards C. G. Darwin gave a solution which is simpler, and which is followed here.

For the hydrogen atom the electron is under the central field from the nucleus only, there being no magnetic field. Thus, if r is the distance of the electron from the nucleus:

$$\mathrm{A}_t = \mathrm{V} = \frac{e}{r},\ \mathrm{A}_x = \mathrm{A}_y = \mathrm{A}_z = 0. \qquad (1)$$

Dirac's equation (1) § 11·4 therefore becomes on account of (7) § 11·3:

$$\left\{\left(p_0 + \frac{e\mathrm{V}}{c}\right) + \alpha_1 p_x + \alpha_2 p_y + \alpha_3 p_z + \alpha_4 m_0 c\right\}\psi = 0. \qquad (2)$$

From the definition of the α's given in (7) § 11·3, and the matrices for the σ's and ρ's, we get

$$\alpha_1 = \rho_1\sigma_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \alpha_2 = \rho_1\sigma_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \tag{3}$$

$$\alpha_3 = \rho_1\sigma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \alpha_4 = \rho_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

According to these 4-rowed matrices, there are four wave-functions $\psi_1, \psi_2, \psi_3, \psi_4$ which can be written in a column

$$\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix}. \tag{4}$$

Since the field is central and we want periodic solutions, we write for p_t, the momentum conjugate to the energy E, its value $\frac{E}{c}$, and for p_x the operator $\frac{h}{2\pi i}\frac{\partial}{\partial x}$, etc.

Thus corresponding to the four rows of the matrices (3), we get from (2) the following four equations for the four wave-functions:

$$\frac{2\pi i}{h}\left(\frac{E + eV}{c} + m_0 c\right)\psi_1 + \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\psi_4 + \frac{\partial}{\partial z}\psi_3 = 0, \tag{5}$$

$$\frac{2\pi i}{h}\left(\frac{E + eV}{c} + m_0 c\right)\psi_2 + \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\psi_3 - \frac{\partial}{\partial z}\psi_4 = 0,$$

$$\frac{2\pi i}{h}\left(\frac{E + eV}{c} - m_0 c\right)\psi_3 + \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\psi_2 + \frac{\partial}{\partial z}\psi_1 = 0,$$

$$\frac{2\pi i}{h}\left(\frac{E + eV}{c} - m_0 c\right)\psi_4 + \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\psi_1 - \frac{\partial}{\partial z}\psi_2 = 0.$$

Equations (5) show that the two functions (ψ_1, ψ_2) form one set of solutions, and the other two functions (ψ_3, ψ_4) form another set.

As the problem is that of a radial force, we must transform to polar co-ordinates (r, θ, ϕ) given by

$$x = r \sin\theta \cos\phi, \quad y = r \sin\theta \sin\phi, \quad z = r \cos\theta. \tag{6}$$

Moreover, since the potential V is a function of r only, being independent of θ and ϕ, the solutions of (5), *i.e.*, the four wave-functions must involve the spherical harmonics S_k^m :

$$S_k^m = P_k^m(\theta) \; e^{im\phi}, \tag{7}$$

where P_k^m are the associated Legendre functions given by

$$P_k^m = (k-m)! \sin^m\theta \left(\frac{d}{d\cos\theta}\right)^{k+m} \frac{(\cos^2\theta - 1)^k}{2^k\, k!}, \tag{8}$$

k being any positive integer, and m being any integer between $-k$ and $+k$ inclusive.

Now if $F(r)$ is any function of r only, then substituting the values of x, y, z, from (6) it is easily calculated that

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) F(r)\, S_k^m = \frac{1}{2k+1} \times$$
$$\left\{\left(\frac{dF}{dr} - \frac{k}{r}F\right) S_{k+1}^{m+1} - (k-m)(k-m-1)\left(\frac{dF}{dr} + \frac{k+1}{r}F\right) S_{k-1}^{m+1}\right\},$$

$$\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) F(r)\, S_k^m = \frac{1}{2k+1} \times$$
$$\left\{-\left(\frac{dF}{dr} - \frac{k}{r}F\right) S_{k+1}^{m-1} + (k+m)(k+m-1)\left(\frac{dF}{dr} + \frac{k+1}{r}F\right) S_{k-1}^{m-1}\right\},$$

$$\frac{\partial}{\partial z} F(r)\, S_k^m = \frac{1}{2k+1} \times$$
$$\left\{\left(\frac{dF}{dr} - \frac{k}{r}F\right) S_{k+1}^{m} + (k+m)(k-m)\left(\frac{dF}{dr} + \frac{k+1}{r}F\right) S_{k-1}^{m}\right\}. \tag{9}$$

Thus considering (5) and (9), we find that if we want each one of the ψ's to involve a single spherical harmonic only, we must write for a trial solution:

$$\begin{aligned} \psi_1 &= -ia_1 f(r)\, S_{k+1}^m, & \psi_2 &= -ia_2 f(r)\, S_{k+1}^{m+1}, \\ \psi_3 &= a_3\, g(r)\, S_k^m, & \psi_4 &= a_4\, g(r)\, S_k^{m+1}. \end{aligned} \tag{10}$$

The same radial function $f(r)$ has to be taken for the two wave-functions ψ_1 and ψ_2 of the first set, and similarly, the same function $g(r)$ has to be taken for ψ_3 and ψ_4. Moreover, $-i$ is introduced in ψ_1 and ψ_2 to make f real. The constants a_1, a_2, a_3, a_4 are introduced so that all the four equations (5) may be satisfied. Thus substituting (10) in (5), and taking account of the relations (9), we obtain

$$
\left.
\begin{aligned}
&\frac{2\pi}{h}\left(\frac{E+eV}{c}+m_0c\right)a_1 f\, S^m_{k+1} + \frac{a_4}{2k+1}\times \\
&\left\{-\left(\frac{dg}{dr}-\frac{k}{r}\right)S^m_{k+1}+(k+m+1)(k+m)\left(\frac{dg}{dr}+\frac{k+1}{r}\right)S^m_{k-1}\right\} \\
&\qquad+\frac{a_3}{2k+1}\left\{\left(\frac{dg}{dr}-\frac{k}{r}g\right)S^m_{k+1}+(k+m)(k-m)\times\right. \\
&\qquad\qquad\left.\left(\frac{dg}{dr}+\frac{k+1}{r}g\right)S^m_{k-1}\right\}=0, \\
&\frac{2\pi}{h}\left(\frac{E+eV}{c}+m_0c\right)a_2 f\, S^{m+1}_{k+1}+\frac{a_3}{2k+1}\times \\
&\left\{\left(\frac{dg}{dr}-\frac{k}{r}g\right)S^{m+1}_{k+1}-(k-m)(k-m-1)\left(\frac{dg}{dr}+\frac{k+1}{r}g\right)S^{m+1}_{k-1}\right\} \\
&\qquad-\frac{a_4}{2k+1}\left\{\left(\frac{dg}{dr}-\frac{kg}{r}\right)S^{m+1}_{k+1}+(k+m+1)(k-m-1)\times\right. \\
&\qquad\qquad\left.\left(\frac{dg}{dr}+\frac{k+1}{r}g\right)S^{m+1}_{k-1}\right\}=0
\end{aligned}
\right\}
\tag{11}
$$

$$
\left.
\begin{aligned}
&\frac{2\pi}{h}\left(\frac{E+eV}{c}-m_0c\right)a_3 g\, S^m_k-\frac{a_2}{2k+3}\times \\
&\left\{-\left(\frac{df}{dr}-\frac{k+1}{r}f\right)S^m_{k+2}+(k+m+2)(k+m+1)\left(\frac{df}{dr}+\frac{k+2}{r}f\right)S^m_k\right\} \\
&\qquad-\frac{a_1}{2k+3}\left\{\left(\frac{df}{dr}-\frac{k+1}{r}f\right)S^m_{k+2}+(k+m+1)(k-m+1)\right. \\
&\qquad\qquad\left.\times\left(\frac{df}{dr}+\frac{k+2}{r}f\right)S^m_k\right\}=0, \\
&\frac{2\pi}{h}\left(\frac{E+eV}{c}-m_0c\right)a_4 g\, S^{m+1}_k-\frac{a_1}{2k+3}\left\{\left(\frac{df}{dr}-\frac{k+1}{r}f\right)S^{m+1}_{k+2}\right. \\
&\qquad\left.-(k-m+1)(k-m)\left(\frac{df}{dr}+\frac{k+2}{r}f\right)S^{m+1}_k\right\} \\
&\qquad+\frac{a_2}{2k+3}\left\{\left(\frac{df}{dr}-\frac{k+1}{r}f\right)S^{m+1}_{k+2}+(k+m+2)(k-m)\right. \\
&\qquad\qquad\left.\left(\frac{df}{dr}+\frac{k+2}{r}f\right)S^{m+1}_k\right\}=0.
\end{aligned}
\right\}
\tag{12}
$$

So far the four constants a_1, etc. have been quite arbitrary. Now we choose them so that the four equations (11) and (12) become as simple as possible. This will be the case if out of the

two spherical harmonics in each equation one disappears, and the remaining equations are consistent.

The two equations (11) show that if we choose

$$a_4(k+m+1)+a_3(k-m)=0, \tag{13}$$

then the term in S_{k-1}^{m} from the first equation, and that in S_{k+1}^{m+1} from the second equation will disappear.

Similarly, the two equations (12) show that if we choose

$$a_1=a_2, \tag{14}$$

then the term in S_{k+2}^{m} from the first equation, and that in S_{k+2}^{m+1} from the second equation will disappear.

The relations (13) and (14) will be satisfied if we write definitely:

$$a_1=1, \quad a_2=1, \quad a_3=k+m+1, \quad a_4=-k+m, \tag{15}$$

because the ratio $a_1 : a_4$ can be incorporated in $f : g$.

Substituting (15) in (11) and (12), and equating the coefficient of the remaining spherical harmonic to zero, we find that the two equations (11) reduce to the same equation

$$\frac{2\pi}{h}\left(\frac{E+eV}{c}+m_0c\right)f+\frac{dg}{dr}-\frac{k}{r}g=0; \tag{16}$$

and the two equations (12) reduce to the same equation

$$-\frac{2\pi}{h}\left(\frac{E+eV}{c}-m_0c\right)g+\frac{df}{dr}+\frac{k+2}{r}f=0. \tag{17}$$

(16) and (17) form a system of two ordinary linear differential equations for the determination of the two unknown functions f and g.

Writing the value of $V=\frac{e}{r}$, and setting for brevity:

$$a^2=\frac{2\pi}{h}\left(\frac{E}{c}+m_0c\right), \quad b^2=\frac{2\pi}{h}\left(-\frac{E}{c}+m_0c\right), \quad \alpha=\frac{2\pi e^2}{ch}, \tag{18}$$

where α is Sommerfeld's fine structure constant (21) § 5·7, the two equations (16) and (17) become:

$$\left(a^2+\frac{\alpha}{r}\right)f+\left(\frac{d}{dr}-\frac{k}{r}\right)g=0, \tag{19}$$

$$\left(\frac{d}{dr}+\frac{k+2}{r}\right)f+\left(b^2-\frac{\alpha}{r}\right)g=0.$$

We transform the dependent variables f, g to f', g' by writing

$$f = e^{-\lambda r} f', \; g = e^{-\lambda r} g', \tag{20}$$

where λ is a constant whose value is to be determined.

Substituting (20) in (19), and dividing out by $e^{-\lambda r}$, we get:

$$\begin{aligned} \left(a^2 + \frac{\alpha}{r}\right) f' + \left(\frac{d}{dr} - \lambda - \frac{k}{r}\right) g' &= 0, \\ \left(\frac{d}{dr} - \lambda + \frac{k+2}{r}\right) f' + \left(b^2 - \frac{\alpha}{r}\right) g' &= 0. \end{aligned} \tag{21}$$

We now try for a solution of (21) in which f' and g' are in the form of the series.

$$\begin{aligned} f' &= a_0 r^\beta + a_1 r^{\beta-1} + a_2 r^{\beta-2} + \cdots, \\ g' &= b_0 r^\beta + b_1 r^{\beta-1} + b_2 r^{\beta-2} + \cdots, \end{aligned} \tag{22}$$

where the constant β, whose value is to be determined, need not be an integer. We have taken series of decreasing powers for f' and g', because these functions have to be finite at $r = 0$.

Substituting (22) in (21), we obtain:

$$\begin{aligned} &a^2 \{a_0 r^\beta + a_1 r^{\beta-1} + a_2 r^{\beta-2} + \cdots\} + \\ &\quad \alpha \{a_0 r^{\beta-1} + a_1 r^{\beta-2} + a_2 r^{\beta-3} + \cdots\} + \\ &\quad \{b_0 \beta r^{\beta-1} + b_1 (\beta - 1) r^{\beta-2} + \cdots\} - \\ &\quad \lambda \{b_0 r^\beta + b_1 r^{\beta-1} + b_2 r^{\beta-2} + \cdots\} - \\ &\quad k \{b_0 r^{\beta-1} + b_1 r^{\beta-2} + b_2 r^{\beta-3} + \cdots\} = 0, \\ &\{a_0 \beta r^{\beta-1} + a_1 (\beta - 1) r^{\beta-2} + a_2 (\beta - 2) r^{\beta-3} + \cdots\} - \\ &\quad \lambda \{a_0 r^\beta + a_1 r^{\beta-1} + a_2 r^{\beta-2} + \cdots\} + \\ &\quad (k + 2) \{a_0 r^{\beta-1} + a_1 r^{\beta-2} + \cdots\} + \\ &\quad b^2 \{b_0 r^\beta + b_1 r^{\beta-1} + b_2 r^{\beta-2} + \cdots\} - \\ &\quad \alpha \{b_0 r^{\beta-1} + b_1 r^{\beta-2} + b_2 r^{\beta-3} + \cdots\} = 0. \end{aligned} \tag{23}$$

We equate the coefficients of successive powers of r in the two equations (23) to zero, and obtain the values of λ, β, $a_0 : b_0$, $a_1 : b_1$, $a_2 : b_2$, etc. Thus, equating the coefficients of r^β to zero, we get

$$a^2 a_0 - \lambda b_0 = 0, \quad b^2 b_0 - \lambda a_0 = 0,$$

from which we find

$$\lambda = \frac{a^2 a_0}{b_0} = \frac{b^2 b_0}{a_0}, \textit{ i.e.}, \quad \frac{b_0}{a_0} = \frac{a}{b} \text{ and } \lambda = ab. \tag{24}$$

We have selected the positive sign for λ, because the functions f and g have to remain finite at $r = \infty$.

Next, equating the coefficients of $r^{\beta-1}$ in (23), we get:

$$a^2 a_1 + \alpha a_0 - \lambda b_1 + (\beta - k)\, b_0 = 0,$$
$$b^2 b_1 - \alpha \beta_0 - \lambda a_1 + (\beta + k + 2)\, a_0 = 0.$$

Substituting the values of $\dfrac{b_0}{a_0}$ and λ from (24), and eliminating both a_1, b_1 from these two equations, we find

$$\beta = -1 + \alpha \frac{a^2 - b^2}{2ab}. \tag{25}$$

Finally, equating the coefficient of $r^{\beta-(s+1)}$ in (23), we obtain:

$$a^2 a_{s+1} + \alpha a_s - \lambda b_{s+1} + (\beta - k - s) b_s = 0,$$
$$b^2 b_{s+1} - \alpha b_s - \lambda a_{s+1} + (\beta + k + 2 - s)\, a_s = 0. \tag{26}$$

Writing the value of $\lambda = ab$, transposing, and multiplying the first equation by b and the second by $-a$, we get:

$$a^2 b a_{s+1} - ab^2 b_{s+1} = -b\alpha a_s - b\,(\beta - k - s)\, b_s,$$
$$a^2 b a_{s+1} - ab^2\, b_{s+1} = a\,(\beta + k + 2 - s)\, a_s - a\alpha b_s.$$

Equating the two right-hand sides, and transposing, we obtain

$$a\, a_s \left\{\beta + k + 2 - s + \alpha \frac{b}{a}\right\} + b\, b_s \left\{\beta - k - s - \alpha \frac{a}{b}\right\} = 0,$$

giving

$$\frac{a_s}{-\dfrac{b}{a}\left\{\beta - k - s - \alpha \dfrac{a}{b}\right\}} = \frac{b_s}{\left\{\beta + k + 2 - s + \alpha \dfrac{b}{a}\right\}} = c_s \quad \text{(say)},$$

where c_s is a new constant to be determined. Then for all $s \geqslant 1$:

$$\left.\begin{aligned} a_s &= c_s \frac{b}{a}\left\{\alpha \frac{a}{b} - \beta + k + s\right\}, \\ b_s &= c_s \left\{\alpha \frac{b}{a} + \beta + k + 2 - s\right\}. \end{aligned}\right\} \tag{27}$$

Substituting these values in any one of the two equations (26), we find the recurrence formula for c_s:

$$ab\,(2s + 2)\, c_{s+1} = -c_s \{(s - \beta - 1)^2 - (j^2 - \alpha^2)\}, \tag{28}$$

where we have written

$$j = k + 1. \tag{29}$$

If we write also

$$\gamma^2 = j^2 - \alpha^2. \tag{30}$$

and so $\gamma = \sqrt{j^2 - \alpha^2}$, supposed positive, we have from (28):

$$2a\, b\,(s+1)\, c_{s+1} = -\, c_s\,(\beta + 1 - s - \gamma)\,(\beta + 1 - s + \gamma). \tag{31}$$

We can find c_1 direct from the value of a_1 or b_1, and then we can determine c_2, c_3, etc. successively from (31). This gives

$$c_s = \frac{(-1)^s}{2^s . s\,!\,(ab)^s}\Big\{(\beta - \gamma + 1)\,(\beta - \gamma) \cdots (\beta - \gamma - s + 2) \times$$
$$(\beta + \gamma + 1)\,(\beta + \gamma) \cdots (\beta + \gamma - s + 2)\Big\}. \tag{32}$$

The functions f', g' are therefore of the type:

$$\begin{aligned} f' &= \sum_s c_s\,\frac{b}{a}\left(\alpha\,\frac{a}{b} - \beta + k + s\right) r^{\beta - s}, \\ g' &= \sum_s c_s\,\left(\alpha\,\frac{b}{a} + \beta + k + 2 - s\right) r^{\beta - s}. \end{aligned} \tag{33}$$

As remraked above, the boundary condition requires that these functions should remain finite for $r = 0$. The series therefore should terminate at a value of s, such that $\beta - s \geqslant 0$ (otherwise we would have a negative power of r).

Supposing that the series terminates at $s = n$, where n is a positive integer or zero, then c_{n+1} should be zero. On account of (31), this would be the case only when the factor

$$\beta + 1 - n - \gamma = 0, \quad \text{or } \beta = \gamma + n - 1. \tag{34}$$

The two equations (25) an (34) give us the eigen-values of the energy parameter. Thus

$$\begin{aligned} \beta + 1 = \gamma + n &= \alpha\,\frac{a^2 - b^2}{2ab}, \\ &= \alpha\,\frac{\mathrm{E}}{\sqrt{m_0{}^2 c^4 - \mathrm{E}^2}}, \end{aligned} \tag{35}$$

on substituting the values of a, b from (18). Solving this for E, we find

$$\begin{aligned} \mathrm{E} &= m_0 c^2 \left\{1 + \frac{\alpha^2}{(n+\gamma)^2}\right\}^{-\frac{1}{2}} \\ &= m_0 c^2 \left\{1 + \frac{\alpha^2}{(n + \sqrt{j^2 - \alpha^2})^2}\right\}^{-\frac{1}{2}}. \end{aligned} \tag{36}$$

This is the same formula as that given by Sommerfeld in (21 *a*) and (21 *b*) § 5·7, if we remember that W = − E, and that for n_1 and n_2 of § 5·7 we have now written n and j respectively.

It is of course possible to get the selection rules on the new theory also, but we shall not go into the details of this question here. They are the same as the rules (30 *a*) § 5 · 7 given by the old quantum theory.

11 · 8. *The Atom Under an External Magnetic Field. The Normal Zeeman Effect.*

In 1896 P. Zeeman discovered experimentally that in an external magnetic field of intensity H, a hydrogen line is split up into three components when viewed at right angles to the field, and two components when viewed along the field. If the frequency of the original line is ν_0, then the frequencies of the two displaced components are $\nu_0 + \Delta\nu$, $\nu_0 - \Delta\nu$, where

$$\Delta\nu = \frac{e\mathrm{H}}{4\pi m_0 c}. \tag{1}$$

Immediately after Zeeman's discovery, and long before the advent of the quantum theory, H. A. Lorenz was able to account for this Zeeman effect by his classical electron theory. This is not surprising if we remark that the constant h, which is characteristic of the quantum theory, does not enter into the displaced frequencies (1). Bohr's quantum theory was equally successful in giving a solution of the problem. In 1916 Sommerfeld and Debye solved it by the method of the Hamilton-Jacobi equation effecting a separation of variables. A little later, Bohr gave a simpler theory with the help of the following theorem due to J. J. Larmor.

Larmor's Theorem.—The effect of a uniform magnetic field H on the motion of an electron under central forces (and some other forces also) is the same, to a first approximation, as if the magnetic field were absent and the whole system had a uniform rotation about an axis parallel to the direction of the field with the angular velocity ψ given by

$$\psi = \frac{e\mathrm{H}}{2m_0 c}.$$

Take the z-axis parallel to the direction of the field H. The components of the mechanical force on the electron at the points x, y, z due to the field are

$$\frac{e\mathrm{H}}{c}\dot{y}, \quad -\frac{e\mathrm{H}}{c}\dot{x}, \quad 0.$$

Let the components of the other force be X, Y, Z. The equations of motion are

$$m_0\ddot{x} = \frac{e}{c}\,\mathrm{H}\dot{y} + \mathrm{X}, \quad m_0\ddot{y} = -\frac{e}{c}\,\mathrm{H}\dot{x} + \mathrm{Y}, \quad m_0\ddot{z} = \mathrm{Z}.$$

Writing $\lambda = \dfrac{e\mathrm{H}}{m_0 c}$, and transposing, we get

$$m_0\,(\ddot{x} - \lambda\dot{y}) = \mathrm{X}, \quad m_0\,(\ddot{y} + \lambda\dot{x}) = \mathrm{Y}, \quad m_0\ddot{z} = \mathrm{Z}. \qquad (a)$$

Now suppose that the electron is free from the action of the magnetic field, but let the system rotate round the z-axis with a uniform angular velocity ψ, then the velocities of the electron are given by

$$u = \dot{x} - y\psi, \quad v = \dot{y} + x\psi, \quad w = \dot{z},$$

and the accelerations are given by $\dot{u} - v\psi$, $\dot{v} + u\psi$, $\dot{w}$, *i.e.*, by

$$\ddot{x} - 2\dot{y}\psi - x\psi^2, \quad \ddot{y} + 2\dot{x}\psi - y\psi^2, \ \ddot{z}.$$

The equations of motion then become

$$m_0\,(\ddot{x} - 2\dot{y}\psi - x\psi^2) = \mathrm{X}, \quad m_0\,(y + 2\dot{x}\psi - y\psi^2) = \mathrm{Y}, \quad m_0\ddot{z} = \mathrm{Z}.$$

Suppose we take $\psi = \dfrac{\lambda}{2} = \dfrac{e\mathrm{H}}{2m_0 c}$, then since c, the velocity of light, is very large, ψ is very small, and we can neglect ψ^2. The equations of motion then become

$$m_0\,(x - 2\dot{y}\psi) = \mathrm{X}, \quad m_0\,(\ddot{y} + 2\dot{x}\psi) = \mathrm{Y}, \quad m_0\ddot{z} = \mathrm{Z}. \qquad (b)$$

Since $2\psi = \lambda$, we see that the equations (a) are identical with the equations (b), and therefore Larmor's theorem is established.

The frequency ω_{L} of this "Larmor Precession" is then given by

$$\omega_{\mathrm{L}} = \frac{1}{2\pi}\cdot\frac{e\mathrm{H}}{2m_0 c} = \frac{e\mathrm{H}}{4\pi m_0 c}. \qquad (2)$$

This additional frequency introduces a new quantum number m in the expression for the energy which explains the splitting up of the spectral lines.

We shall now give a detailed solution of the problem on the new mechanics, following the treatment given by Dirac.

We take a uniform magnetic field of intensity $\vec{\mathrm{H}}$, acting in the direction of the z-axis, so that

$$\mathrm{H}_x = 0, \quad \mathrm{H}_y = 0, \quad \mathrm{H}_z = \mathrm{H}, \quad \mathrm{E}_x = 0, \quad \mathrm{E}_y = 0, \quad \mathrm{E}_z = 0. \qquad (3)$$

Consequently, the scalar and vector potentials are

$$A_t = V = \frac{e}{r}, \quad A_x = -\frac{1}{2}Hy, \quad A_y = +\frac{1}{2}Hx, \quad A_z = 0. \tag{4}$$

In equation (9) § 11 · 6 we showed that the effect of the spin of the electron moving under an external electromagnetic field is to increase the potential energy by

$$\frac{1}{2m_0}\frac{h}{2\pi}\frac{e}{c}(\sigma_1 H_x + \sigma_2 H_y + \sigma_3 H_z) + \frac{1}{2m_0}\frac{ih}{2\pi}\frac{e}{c}$$
$$\rho_1(\sigma_1 E_x + \sigma_2 E_y + \sigma_3 E_z),$$

which, in the present case, becomes, on account of (3) :

$$\frac{1}{2m_0}\cdot\frac{h}{2\pi}\frac{e}{c}\sigma_3 H. \tag{5}$$

Since the effect of the relativistic variation of the mass of the electron with its velocity is much smaller compared to the effect of the external magnetic field, we shall neglect the relativity effect, and take the classical Hamiltonian (17) of § 10 · 1 (4),

$$-eA_t + \frac{1}{2m_0}\left\{\left(p_x + \frac{e}{c}A_x\right)^2 + \left(p_y + \frac{e}{c}A_y\right)^2 + \left(p_z + \frac{e}{c}A_z\right)^2\right\}.$$

Adding the potential energy given by (5), and substituting the values from (4), we have for the Hamiltonian of the problem :

$$H' = \frac{1}{2m_0}\left\{\left(p_x - \frac{e}{2c}Hy\right)^2 + \left(p_y + \frac{e}{2c}Hx\right)^2 + p_z^2\right\}$$
$$-\frac{e^2}{r} + \frac{ehH}{4\pi m_0 c}\sigma_3. \tag{6}$$

If the magnetic field is not too large, the terms involving H^2 can be neglected, and (6) reduces to

$$H' = \frac{1}{2m_0}(p_x^2 + p_y^2 + p_z^2) - \frac{e^2}{r} + \frac{eH}{2m_0 c}(xp_y - yp_x) + \frac{eH}{2m_0 c}\frac{h}{2\pi}\sigma_3. \tag{7}$$

From (1) § 6 · 8 (1) we know that the component m_z of the orbital angular momentum is $m_z = xp_y - yp_x$. The Hamiltonian (7) can therefore be wirtten :

$$H' = \frac{1}{2m_0}(p_x^2 + p_y^2 + p_z^2) - \frac{e^2}{r} + \frac{eH}{2m_0 c}\left(m_z + \frac{h}{2\pi}\sigma_3\right). \tag{8}$$

The Hamiltonian of the hydrogen atom for no external field is simply $\frac{1}{2m_0}(p_x^2 + p_y^2 + p_z)^2 - \frac{e^2}{r}$, so that the extra terms due

to the magnetic field are $\frac{eH}{2m_0c}\left(m_z + \frac{h}{2\pi}\sigma_3\right)$. Now in equations (7) and (8) of § 6 · 8 (1) it has been proved that each component of the orbital angular momentum, and therefore m_z, commutes with r and with $p_x^2 + p_y^2 + p_z^2$. σ_3, being a diagonal matrix, commutes, of course, with every matrix. The additional terms $\frac{eH}{2m_0c}\left(m_z + \frac{h}{2\pi}\sigma_3\right)$ commute, therefore, with the total Hamiltonian (8), and are thus constants of the motion, as shown in Chapter VI. Writing

$$H_0 = \frac{1}{2m_0}(p_x^2 + p_y^2 + p_z^2) - \frac{e^2}{r}, \tag{9}$$

and

$$H_1 = \frac{eH}{2m_0c}\left(m_z + \frac{h}{2\pi}\sigma_3\right), \tag{10}$$

the Hamiltonian (8) can be written

$$H' = H_0 + H_1, \tag{11}$$

where H_0 is the Hamiltonian for no external field, and H_1 is the additional term due to the field. As shown above, H_1 commutes with H', and is a constant of the motion. Therefore, the eigen-functions for the Hamiltonian H' will be those eigen-functions of H_0 which are also eigen-functions of H_1, since

$$H'\psi = H_0\psi + H_1\psi. \tag{12}$$

Moreover, if E, E_0, E_1 are the eigen-values corresponding H', H_0, H_1 respectively, then from (11) we have

$$E = E_0 + E_1, \tag{13}$$

showing that the presence of the magnetic field causes the stationary states to differ only in the value of the energy, and not in the form of the eigen-functions. This is the wave-mechanical analogue of Larmor's theorem.

Now the eigen-values of the orbital angular momentum m_z are given by the rule that the angular momentum is an integral multiple m of $\frac{h}{2\pi}$:

$$m_z = m\frac{h}{2\pi}, \quad (m \text{ integer}). \tag{14}$$

From the matrix for σ_3, its eigen-values are easily found to be $\sigma_3' = \pm 1$, because the eigen-values of a diagonal matrix are the terms in the leading diagonal.

Thus we find from (10) :

$$E_1 = \frac{eH}{2m_0c}\left(m\frac{h}{2\pi} + \frac{h}{2\pi}\sigma_3'\right). \tag{15}$$

The selection rules for m_z now require that only those transitions are possible in which the magnetic quantum number m changes to $m + \Delta m$ such that

$$\Delta m = -1, 0, +1. \tag{16}$$

Since σ_3 commutes with every matrix, and also with the electric displacement, it will not change at all. We have therefore the result that in a transition E_1 changes to $E_1 + \Delta E_1$, where

$$\Delta E_1 = \frac{eH}{2m_0c}\cdot\frac{h}{2\pi}\Delta m. \tag{17}$$

Considering the Zeeman effect for a particular hydrogen line $\nu_0 = \frac{E_0}{h}$, we see on account of (13) and (17) that it is split up into the components $\nu_0 + \Delta\nu$, given by

$$\Delta\nu = \frac{\Delta E_1}{h} = \frac{eH}{4\pi m_0c}\Delta m, \tag{18}$$

where Δm has the three values (16). These displacements (18) are in agreement with experiments.